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Investigations on a bi-chiral scalar field

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Abstract : Chiral boson fields in (1+1) dimensions obey a first order equation of motion. A second order equation which classically describes two chiral boson fields of the same chirality leads, however, to an unacceptable quantum theory.

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A real massless boson field satisfies the equation

$$\Box\phi = 0. \tag{1}$$

In (1+1) dimensions, this means that the product of the two lightcone components of the momentum vanishes. There are two ways of solving this equation and one may regard the massless boson field as being the assembly of two *chiral* boson fields, one with the + component of the momentum vanishing and the other with the component vanishing. Thus a chiral boson field satisfies an equation

$$\partial_{\pm}\phi = 0, \tag{2}$$

where $$\partial_{+} = \partial_0 + \partial_1. \tag{3}$$

The equation of motion is a first order one and this makes the construction of a quantum theory somewhat nontrivial. Siegel [1] started with the Lagrangian for a massless scalar field and tried to make it chiral by imposing a constraint. A term of degree three involving the square of $\partial_{\pm}\phi$ and an auxiliary field had to be introduced. Subsequently, Floreanini and Jackiw [2] discovered a quadratic Lagrangian involving no auxiliary field, but this Lagrangian does not possess explicit Lorentz invariance. Of course, the *theory* is Lorentz invariant [2].

If a lightcone component of the momentum vanishes, its square vanishes too, so one might be tempted to think that a second order equation would also describe a chiral boson field. But the second order equation is a weaker restriction on the field and it is more natural to expect something similar to a full field degree of freedom rather than the half degree that a chiral boson field is. As the boson field is definitely chiral, one is then led to expect roughly a pair of chiral boson fields of the same chirality, in contrast to a scalar field which is equivalent to two chiral boson fields of opposite chiralities. In this note, we analyse in detail the theory underlying the second order equation. The surprising result is that one does have two chiral boson fields but in quite a complicated way, and in fact the corresponding quantum theory is not well defined.

We start with the Lagrangian density

$$L = \frac{1}{2}(\partial_+\phi)^2. \tag{4}$$

This Lagrangian density is not manifestly Lorentz invariant, but Lorentz invariance is maintained, as is clear from the Euler-Lagrange equation

$$\partial_+^2\phi = 0. \tag{5}$$

The eq. (5) requires ϕ to be of the form

$$\phi = \phi_0(x^-) + \frac{x^+}{2}\phi_1(x^-), \tag{6}$$

where $$x^\pm = x^0 \pm x^1. \tag{7}$$

This indicates the existence of two chiral boson fields of the same chirality, *i.e.*, travelling in the same direction (right).

The momentum corresponding to the field ϕ is given by

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \partial_+\phi = \phi_1(x^-). \tag{8}$$

The Hamiltonian density is obtained by the Legendre transformation

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \tag{9}$$

and the total Hamiltonian is found to be

$$H = \int dx\left[-\phi_1\phi_0' + \frac{1}{4}\phi_1^2\right], \tag{10}$$

where an integration by parts has been carried out : the field is assumed to vanish sufficiently fast at spatial infinity. This Hamiltonian is not hermitian. To get a hermitian Hamiltonian one must take

$$H = \int dx\left[-\frac{1}{2}\phi_1\phi_0' - \frac{1}{2}\phi_0'\phi_1 + \frac{1}{4}\phi_1^2\right]. \tag{11}$$

The components of the energy momentum tensor $T_{\mu\nu}$ may also be calculated and are given by

$$T_{00} = \frac{1}{2}\left(\dot{\phi}^2 - \phi'^2\right), \tag{12}$$

$$T_{01} = \dot{\phi}\phi' + \phi'^2, \tag{13}$$

$$T_{11} = \frac{1}{2}\left(\dot{\phi}^2 - \phi'^2\right), \tag{14}$$

$$T_{10} = -\dot{\phi}\phi' - \dot{\phi}^2. \tag{15}$$

The conservation of $T_{\mu\nu}$ is easy to see.

The Fourier expansion of ϕ may be taken as

$$\phi_0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{dk}{|k|}\, a(k)\, e^{ikx^-} \tag{16}$$

$$\phi_1 = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk|k|b(k)\, e^{ikx^-} \tag{17}$$

where the hermiticity of ϕ requires that

$$a^\dagger(k) = a(-k), \quad b^\dagger(k) = -b(-k). \tag{18}$$

The quantization condition is

$$\left[\phi(x), \pi(y)\right]_{x^0=y^0} = i\delta(x^1 - y^1), \tag{19}$$

which is equivalent to

$$\left[\phi_0(x^-), \phi_1(y^-)\right] = i\delta(x^- - y^-), \tag{20}$$

and hence to

$$\left[a(k), b(l)\right] = \delta(k+l). \tag{21}$$

All other commutators are zero. Now, we define and annihilation operator and a creation operator for a real parameter λ by

$$c(k) = \frac{\lambda^{-1}a(k) - \lambda|k|b(k)}{\sqrt{2|k|}}, \quad c^\dagger(k) = \frac{\lambda^{-1}a(-k) + \lambda|k|b(-k)}{\sqrt{2|k|}}. \tag{22}$$

The commutation relations satisfied by $c(k)$ and $c^\dagger(k)$ are

$$\left[c(k), c^\dagger(l)\right] = \delta(k-l), \left[c(k), c(l)\right] = 0. \tag{23}$$

Substituting the values of ϕ_0 and ϕ_1 in the expression of H, one obtains

$$H = \int_{-\infty}^{+\infty} dk\, \frac{1}{2}\left[ka(-k)b(k) + kb(k)a(-k) - \frac{1}{2}k^2 b(k)b(-k)\right]. \tag{24}$$

which can also be written as

$$H = -\frac{1}{2}\int_{-\infty}^{+\infty} dk\, k\left[c^{\dagger}(k)c(k) + c(k)c^{\dagger}(k)\right] + \frac{1}{8\lambda^2}\int_{-\infty}^{+\infty} dk\, |k| \left[c^{\dagger}(k)c(k) + c(k)c^{\dagger}(k) - c(k)c(-k) - c^{\dagger}(-k)c^{\dagger}(k)\right]. \tag{25}$$

The energy of the system depends on the modes k that are excited. As k runs from $-\infty$ to $+\infty$, the energy has no lower (or upper) bound, as may be seen by making λ large. So a vacuum cannot be defined by the condition $c(k)\,|\,0> = 0$.

To compare with the Floreanini Jackiw form of Lagrangian for a right moving chiral boson field, let'us start with the Siegel action for a right moving chiral boson field :

$$S = \frac{1}{2}\int d^2x\left[\partial_+\phi\partial_-\phi + \lambda(\partial_+\phi)^2\right]. \tag{26}$$

Starting from the action (26), a Lagrangian density for right moving chiral boson field can be derived where Lorentz invariance is not manifested [2] :

$$\mathcal{L} = -\dot{\phi}\phi' - \phi'^2. \tag{27}$$

So it is sufficient if it can be shown that the energy corresponding to the Lagrangian density (27) is bounded. The canonical momentum corresponding to the field ϕ is

$$\pi_\phi = -\phi'. \tag{28}$$

$\pi_\phi + \phi' = 0$ is a second class constraint itself. So Dirac's prescription for quantization of second class constrained system has to be followed. The Dirac bracket is found out to be

$$\left[\phi(x),\, \phi(y)\right]_{Dirac} = \frac{1}{4}\varepsilon(x-y). \tag{29}$$

The reduced Hamiltonian is

$$H_r = \int dx\phi'^2. \tag{30}$$

Using eqs. (29) and (30), one can show that the field ϕ satisfies the equation

$$\partial_+\phi = 0. \tag{31}$$

Since the Hamiltonian (30) is positive definite, the energy of this system is bounded below, whereas the energy of the system described by the Hamiltonian (10) is not so bounded. Thus, the system of bichiral bosons, although classically interesting, does not correspond to any quantum field theory.

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References

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